JOURNAL OF MULTIVARIATE ANALYSIS 2, 230–235 (1972)

Characteristic Functions and Bernoulli Numbers

DANIEL DUGUÉ*

Catholic University of America
and Institut de Statistique de l'Université de Paris

Communicated by E. Lukacs

Fredholm's integral equation theory and Mittag–Leffler expansion is used for getting characteristic functions and three ways of expressing the Bernoulli numbers.

This is a supplement to a paper which the author presented at the Washington session of the I.S.I. in August, 1971.

Let $X(t)$ be a normal stochastic process defined for $0 \leqslant t \leqslant 1$ with a covariance function $c(t_1, t_2)$ and a mean value $E(X(t)) = f(t)$. We shall get a very simple expression for the characteristic function of $\int_0^1 X^2(t)\,dt$. This is well known if $f(t) = 0$ and especially in the case of the Wiener–Lévy and von Mises–Smirnoff process. It is somewhat more complicated if $f(t) \neq 0$. As usual $\int_0^1 X^2(t)\,dt$ will be approximated through the Riemann sum

$$\frac{1}{s}\left[X^2\left(\frac{1}{s}\right) + \cdots + X^2\left(\frac{s}{s}\right)\right].$$

Let ${}^tV$ be the vector $[f(1/s) \cdots f(s/s)]$ and ${}^tX$ the vector $[X(1/s) \cdots X(s/s)]$. The covariance matrix of X will be $C_s = ((c(i/s, j/s))$. And we have

$$E\left(\exp iu \int_0^1 X^2(t)\,dt\right)$$
$$= \lim_{s\to\infty}\left[\frac{1}{(2\pi)^{s/2}\sqrt{\det C_s}} \int \exp\left[iu\,\frac{{}^tXX}{s} - \frac{1}{2}\,{}^t(X - V)\,C_s^{-1}(X - V)\right] dX.\right.$$

Received December 8, 1971.

AMS 1970 subject classifications: Primary 62G15.

Key words and phrases: Characteristic functions; Bernoulli numbers; Gaussian process.

* This work was done under the National Science Foundation grant number NSF-GP-22585 while the author was a visiting professor at Catholic University.

This equality holds if $c(t_1, t_2)$ is sufficiently regular. We have

$$i\frac{u}{s}\,{}^tXX = i\frac{u}{s}\,{}^t(X-V)(X-V) + i\frac{u}{s}\,{}^tVV + 2i\frac{u}{s}\,{}^tV(X-V)$$

and so we get

$$E\left[\exp i\frac{u}{s}\,{}^tXX\right]$$

$$= \frac{e^{i(u/s)\,{}^tVV}}{(2\pi)^{s/2}\sqrt{\det C_s}}$$

$$\times \int \exp\left[2i\frac{u}{s}\,{}^tV(X-V) - \frac{1}{2}\,{}^t(X-V)\left[C_s^{-1} - \frac{2iu}{s}I_s\right](X-V)\right] d(X-V)$$

$$= e^{i(u/s)\,{}^tVV}\left[\det\left[I_s - \frac{2iu}{s}C\right]\right]^{-1/2} \exp - \frac{4u^2}{2s}\,{}^tV\left[C_s^{-1} - \frac{2iu}{s}I_s\right]^{-1} V. \quad (1)$$

If $f(t)$ is Riemann square integrable $(1/s)\,{}^tVV$ tends towards $\int_0^1 f^2(t)\,dt$. If $c(t_1, t_2)$ is sufficiently regular $\det[I_s - (2iu/s)\,C_s]$ has a limit equal to

$$\prod_{k=1}^{+\infty}\left(1 - \frac{2iu}{\lambda_k}\right)$$

where the $\{\lambda_k\}$ are the eigenvalues of $\varphi(t_1) = \lambda\int_0^1 f(t_1, t_2)\,\varphi(t_2)\,dt_2$. This limit has a very simple form in the Wiener–Lévy case ($\cos\sqrt{2iu}$) and in the von Mises–Smirnoff case ($\sin\sqrt{2iu}/\sqrt{2iu}$). As $c(t_1, t_2)$ is symmetric and positive definite, λ_k is real and nonnegative. We have

$$\frac{2u^2}{s^2}\,{}^tV\left[C_s^{-1} - \frac{2iu}{s}I_s\right]^{-1} V = \frac{2u^2}{s^2}\,{}^tVC_s\left[I_s - \frac{2iu}{s}C_s\right]^{-1} V$$

$$= \frac{2u^2}{s^2}\sum_{n=0}^{+\infty}\left[\frac{2iu}{s}\right]^n {}^tVC_s^{n+1}V$$

$$= 2u^2\sum_{n=0}^{+\infty}(2iu)^n\,\frac{{}^tVC_s^{n+1}V}{s^{n+2}}\,.$$

Let A_s be an $s \times s$ matrix such that $(1/s)\,A_s{}^tA_s = I_s$ and diagonalize C_s. Then $(1/s)\,C_s = (1/s)\,{}^tA_sD_sA_s$, where D_s is a diagonal matrix with elements tending to λ_k^{-1}, so that ${}^tVC_s^{n+1}V/s^{n+2}$ tends towards $(1/s)\,{}^tV{}^tA_sD_s^{n+1}(1/s)\,A_sV$.

The limit of this scalar is equal to:

$$\sum_{k=1}^{+\infty} a_k^2\,\frac{1}{\lambda_k^{n+1}} \qquad \text{with} \quad a_k = \int_0^1 \varphi_k(u)\,f(u)\,du,$$

where φ_k is the k-th eigenfunction of $c(t_1, t_2)$, i.e.,

$$\varphi_k(u) = \lambda_k \int_0^1 c(u, t)\, \varphi_k(u)\, du \qquad \text{with} \quad \int_0^1 \varphi_k^2(u)\, du = 1.$$

Therefore in formula (1) the exponent will be

$$-2u^2 \sum_{n=0}^{+\infty} (2iu)^n \left[\sum_{k=1}^{+\infty} \frac{a_k^2}{\lambda_k^{n+1}}\right] = -2u^2 \sum_{k=1}^{+\infty} a_k^2 \frac{1}{2iu} \left[\frac{1}{1 - \dfrac{2iu}{\lambda_k}} - 1\right].$$

Hence the characteristic function of $\int_0^1 X^2(t)\, dt$ becomes

$$\left[\prod_{k=1}^{\infty} \left(1 - \frac{2iu}{\lambda_k}\right)\right]^{-1/2} \exp \left\{ iu \int_0^1 f^2(t)\, dt + iu \sum_{k=1}^{+\infty} a_k^2 \left[\frac{1}{1 - \dfrac{2iu}{\lambda_k}} - 1\right] \right\}.$$

If the set of eigenfunctions is a set of completely orthogonal functions

$$\int_0^1 f^2(t)\, dt = \sum_{k=1}^{+\infty} a_k^2$$

and the exponent is

$$\sum_{k=1}^{+\infty} a_k^2 \frac{iu}{1 - \dfrac{2iu}{\lambda_k}} = \frac{1}{2} \sum_{k=1}^{+\infty} a_k^2 \lambda_k \left[\frac{1}{1 - \dfrac{2iu}{\lambda_k}} - 1\right].$$

With this form we see immediately that

$$\exp \sum_{k=1}^{+\infty} a_k^2 \frac{iu}{1 - \dfrac{2iu}{\lambda_k}}$$

is the limit of characteristic functions of an indefinitely divisible law. And so the law of $\int_0^1 X^2(t)\, dt$ is indefinitely divisible in the general case. The Taylor expansion of the exponent is

$$\frac{1}{2} \left[(2iu) \sum_{k=1}^{+\infty} a_k^2 + \cdots + (2iu)^n \sum_{k=1}^{+\infty} \frac{a_k^2}{\lambda_k^{n-1}} + \cdots\right]$$

$\sum_{k=1}^{\infty} a_k^2 = \int_0^1 f^2(t)\, dt$ if $\{\varphi_k\}$ is complete. In this case,

$$c(t_1, t_2) = \sum_{k=1}^{\infty} \frac{\varphi_k(t_1)\, \varphi_k(t_2)}{\lambda_k}$$

and we see that

$$f_{(1)}(t_1) = \int_0^1 f(t_1, t_2) f(t_2)\, dt_2 = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k} \varphi_k(t_1)$$

and

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k} = \int_0^1 f(t) f_{(1)}(t)\, dt,$$

and so forth,

$$f_{(n)}(t_1) = \int_0^1 f(t_1, t_2) f_{(n-1)}(t_2)\, dt_2 = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^n} \varphi_k(t_1)$$

and

$$\sum_{k=1}^{\infty} \frac{a_k^2}{\lambda_k^n} = \int_0^1 f(t) f_{(n)}(t)\, dt.$$

So any exponent of this form will lead to a characteristic function of an indefinitely divisible law.

Obviously, if $\sum_{k=1} a_k^2 \lambda_k$ is convergent,

$$\left[\sum_{k=1}^{+\infty} a_k^2 \lambda_k\right]^{-1} \sum_{k=1}^{+\infty} a_k^2 \lambda_k \left[\frac{1}{1 - \dfrac{2iu}{\lambda_k}}\right]$$

is a characteristic function. Let us suppose we have even $\sum_{k=1}^{+\infty} a_k^2 \lambda_k^2 < \infty$. Then $\sum_{k=1}^{+\infty} a_k \lambda_k \varphi_k(t)$ is mean-square convergent towards $f_{(-1)}(t)$ with

$$\sum_{k=1}^{\infty} a_k \lambda_k \varphi_k(t) = f_{(-1)}(t).$$

We have

$$f(t_1) = \int_0^1 c(t_1, t_2) f_{(-1)}(t_2)\, dt_2 \qquad \text{and} \qquad \sum_{k=1}^{\infty} a_k^2 \lambda_k = \int_0^1 f_{(-1)}(t) f(t)\, dt.$$

Therefore we have the following theorem:

THEOREM. *The Taylor series*

$$1 + \sum_{n=1}^{\infty} (2iu)^n \left[\int_0^1 f_{(-1)}(t) f(t)\, dt\right]^{-1} \int_0^1 f(t) f_{(n-1)}(t)\, dt$$

is a characteristic function provided that $f_{(-1)}(t)$ is an arbitrary square integrable function and if $c(t_1, t_2)$ is a positive definite kernel with a complete set of eigenfunctions.

In the quoted paper, I gave the example $c(t_1, t_2) = \min(t_1, t_2)$ and $f(t) = m$, so the characteristic function is

$$[\cos \sqrt{2iu}]^{-1/2} \exp \tfrac{1}{2} m^2 \sqrt{2iu}\ tg\ \sqrt{2iu}. \tag{A}$$

Using the following definition of the Bernoulli numbers:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

the expansion of $\sqrt{2iu}\ tg \sqrt{2iu}$ is

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k}-1)}{(2k)!} B_{2k}(2iu)^k.$$

And so we have

$$B_{2k} = (-1)^{k-1} \frac{(2k)!}{2^{2k}(2^{2k}-1)} \int_0^1 f_{(n-1)}(t)\, dt, \tag{α}$$

with $f_{(n)}(t) = \int_0^1 \min(t, x) f_{(n-1)}(x)\, dx$ and $f(x) = 1$. Now I shall give two other examples.

If we consider the Wiener–Lévy process $X(t)$ with $E(X(t)) = m$ and if we take $Y(t) = X(t) - t(X(1) - X(0))$ we have $c(t_1, t_2) = \min(t_1, t_2) - t_1 t_2$ and $f(t) = m$. It is very easy to see that

$$E\left(\exp iu \int_0^1 Y^2(t)\, dt\right) = \left[\frac{\sqrt{2iu}}{\sin \sqrt{2iu}}\right]^{1/2} \exp 2m^2 \sqrt{i\frac{u}{2}}\, tg \sqrt{i\frac{u}{2}}\,. \tag{B}$$

With this result we see that

$$E\left[\int_0^1 Y^2(t)\, dt\right] = \frac{1}{6} + m^2 \qquad \text{and variance of} \quad \int_0^1 Y^2(t)\, dt = \frac{1}{45} + \frac{m^2}{3}\,.$$

From formula (A)

$$E\left[\int_0^1 X^2(t)\, dt\right] = \frac{1}{2} + m^2 \qquad \text{and variance of} \quad \int_0^1 X^2(t)\, dt = \frac{1}{3} + \frac{4}{3} m^2\,.$$

Therefore $\int_0^1 Y^2(t)\, dt - \frac{1}{6}$ is a more efficient estimation of m^2 than $\int_0^1 X^2(t)\, dt - \frac{1}{2}$.

In order to have cotg in the exponent, we have just to take a $Z(t)$ process with

$$c(t_1, t_2) = \min(t_1 t_2) - t_1 t_2 \qquad \text{and} \qquad f(t) = m(1 - t),$$

and so we have

$$E\left(\exp iu \int_0^1 Z^2(t)\, dt\right) = \left[\frac{\sqrt{2iu}}{\sin \sqrt{2iu}}\right]^{1/2} \exp \frac{m^2}{2} (1 - \sqrt{2iu} \operatorname{cotg} \sqrt{2iu}). \quad \text{(C)}$$

Through (B) and (C) it is easy to get a new expression for the Bernoulli numbers which is similar to (α).